A General Expression for the Correlation of Rates of Transfer and Other Phenomena

The expression $Y = (1 + Z^n)^{1/n}$ where Y and Z are expressed in terms of the solutions for asymptotically large and small values of the independent variable is shown to be remarkably successful in correlating rates of transfer for processes which vary uniformly between these limiting cases. The arbitrary exponent n can be evaluated simply from plots of Y versus Z and Y/Z versus $1/Z$. The expression is quite insensitive to the choice of n and the closest integral value can be chosen for simplicity. The process of correlation can be repeated for additional variables in series. Illustrative applications are presented only for flow, conduction, forced convection, and free convection, but the expression and procedure are applicable to any phenomenon which varies uniformly between known, limiting solutions.

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SCOPE

The limiting solutions for large and small values of the independent variables, and parameters such as the Reynolds number, the Prandtl number, and time are known for many transfer processes such as laminar and turbulent flow through porous media, laminar forced convection, and thermal conduction. Solutions for intermediate cases are generally not in closed form.

The use of a simple equation containing one arbitrary constant is proposed for interpolation between these limiting solutions. The constant can be evaluated from one or more experimental or theoretical values. The equation itself has been used previously for correlation, but its general utility has not been recognized and a systematic method of evaluating the constant has not been proposed. The general applicability of the equation and the procedure of correlation are illustrated for a number of processes of flow and heat transfer. The expression and procedure appear however to be applicable to any phenomenon which varies uniformly between known, limiting solutions.

The proposed expression is useful for evaluating as well as summarizing experimental data and values obtained from computer solutions. It is convenient for design purposes, particularly since all of the correlations have the same form.

CONCLUSIONS AND SIGNIFICANCE

The equation $Y = (1 + Z^n)^{1/n}$, where n is an arbitrary constant and Y and Z are defined in terms of the limiting solutions for large and small values of the independent variable, is remarkably successful in representing data for which the dependent variable varies uniformly from small to large values of the independent variable. The majority of transfer processes fall in this category. Examples are the dependence of the pressure drop on the rate of flow through porous media and the dependence of the Nusselt number on the Prandtl number in laminar free convection. Processes which have an irregular dependence such as the pressure drop through a pipe on the Reynolds number in the region of transition from laminar to turbulent flow cannot of course be successfully represented.

For processes in which the dependence on the independent variable increases as the independent variable increases, Y is chosen as $y(z)/y(z \to 0)$ and Z as $y(z \to \infty)/y(z \to 0)$ where $y(z \to 0)$ and $y(z \to \infty)$ are the asymptotic solutions for small and large z, respectively. For processes in which the dependent variable decreases as the independent variable increases, Y is chosen instead as $y(z \to 0)/y(z)$ and Z as $y(z \to 0)/y(z \to \infty)$. Since $y(z \to 0)$ and $y(z \to \infty)$ cannot both be constants Z is a function of z. The correlation is equivalent to the nth-order sum of the asymptotic solutions and is proportional to the nth root mean value. For example, if $n = 2$ the correlation is equivalent to $\sqrt{2}$ times the root mean square of the asymptotic values.

Experimental data and computer solutions for representative processes appear to follow the functional relationship provided by this equation very closely even though there is no apparent theoretical justification for its use. This success apparently is a result of the great similarity of the different rate processes when expressed in this canonical form and of the limited variance from the asymptotic solutions. The equation or a plot of Y versus Z can thus be considered to be a general representation of all rate processes with uniform behavior, individual processes being characterized by n alone.

The arbitrary exponent n can be evaluated by comparing experimental data or computed solutions with curves of Y versus Z or Y/Z versus $1/Z$ for fixed values of n. The correlations for typical processes prove to be relatively insensitive to the choice of n and in many instances an integral value can be chosen without serious error. Conversely such plots provide a very sensitive test of the consistency, precision, and accuracy of experimental data and computer solutions. In some cases the equation can be applied in series (with different Y, Z, and n) for two or more independent variables or parameters. In other cases n can be correlated as a function of the additional variables.

The intersection of the two limiting solutions, which occurs at $Z = 1$, defines a central value for the independent variable. This central value is different for different processes. For example, the central value of the Prandtl number is 0.492 for free convection from a vertical isothermal plate, but is 0.0467 for forced convection from an isothermal plate. This central value is optimal for experimentation or numerical solution insofar as correlation is concerned since the related deviation of Y from the limiting solutions is greatest. That is, the value of n is most sensitive to experimental or computed values at or near $Z = 1$.

Flow through porous media, the free fall of a sphere, laminar free convection from an isothermal vertical plate, laminar forced convection from a flat plate, and transient thermal conduction to an insulated semi-infinite region are examples of applications in which the supporting data, series solutions, and computer solutions are quite consistent and dependable and in which precise correlations are obtained by this procedure. Laminar forced convection in a tube is illustrative of applications in which there is a choice between different asymptotic solutions and in which the correlation is not quite so successful. This latter example is included to indicate that the development of the best correlation is not necessarily routine and may require ingenuity and judgement despite the simplicity of the general expression.

Although these examples are for simple processes of flow and heat transfer, the equation and procedure appear to be applicable to mass transfer, to more complex processes of transfer, and even to physical relationships other than transfer.

Tabulations of experimental or theoretical values are essential for the development of correlations. The common practice of presenting such results only in graphical form is unfortunate in this respect. On the other hand, tabulations as well as theoretical solutions in series and integral form are generally inconvenient for direct use in design and analysis. Log-log or semilog plots generally show the forest but not the trees; deviations are compressed and effectively disguised. In printed form such plots can seldom be read with sufficient accuracy for applications. Arithmetric plots are usually limited in utility to a narrow range of one of the variables. By contrast, the special graphical representations advocated herein display only deviations from the limiting solutions and emphasize the scatter of individual points. Hence they provide a better basis for critical analysis of the individual points and of the success of proposed correlations. Even these plots are not very convenient for direct use. Empirical equations are generally the most convenient for the application of experimental and complex theoretical results. The form of the empirical equation proposed herein has the advantage of simplicity, generality, inherent accuracy, and convergence to theoretical solutions in the limits.

Solutions in closed form have been developed for many transfer processes for asymptotically large and small values of time, Pr, Sc, Re, Gz, Gr, etc. Solutions for intermediate values have generally been accomplished only in the form of infinite series, definite integrals, and tabulations of numerical integrations. These intermediate solutions are often inconvenient to use. For example, the solution derived by Graetz (1885) for convective heat transfer in fully developed laminar flow in a tube following a step-change in wall-temperature converges very slowly for large Gz. In most cases such solutions for intermediate values can be evaluated only at a series of discrete values of the independent variable, and a graphical representation or an empirical correlation of these values is desirable for interpolation. Many different expressions, including power series, have been used for this purpose, but apparently no general expression has been proposed.

Empirical expressions for the entire range have been constructed in some cases from the limiting solutions for large and small values alone. For example, the equation developed by Ergun (1952) for the pressure drop in flow through randomly packed beds of uniformly sized spheres consists of the simple sum of the empirical correlations for the two limiting cases of purely laminar flow and purely turbulent flow. In most cases additional terms and arbitrary constants have been found necessary to fit the intermediate values. For example, Le Fevre (1956) proposed the following empirical equation for laminar free convection from a vertical, isothermal plate:

$$Nu_x = \frac{3}{4}\left[\frac{GrPr^2}{2.435 + 4.884\,Pr^{1/2} + 4.952\,Pr}\right]^{1/4} \quad (1)$$

where the first and third terms in the denominator of the bracket correspond to the asymptotic solutions for small and large Pr, respectively, and the coefficient of $Pr^{1/2}$ was chosen to fit the intermediate values of the numerical solution of Ostrach (1953).

Acrivos (1961) utilized Equation (7) below with the variables interpreted as in Equation (16) for representation of the effect of Pr and Sc on convection in several laminar boundary layer flows. He recommended specific, integral values of n for these different situations but did not discuss the method of evaluation of the arbitrary exponent n or the accuracy of the representation. Many others have utilized this expression but generally in a more restricted sense.

The development of a general expression and procedure for correlation in terms of asymptotic solutions for large and small values of the independent variable and one arbitrary constant is outlined in the following paragraphs. The general expression is the same as that used by Acrivos but its application is extended to other variables and processes and to increasing as well as decreasing dependence on the independent variable. Principle attention is given to the evaluation of the arbitrary exponent and to the choice of the most appropriate asymptotic solutions.

DEVELOPMENT

Case A. Increasing Dependence

When the power of the independent variable is greater at the higher limit, that is, when

$$y \to Az^p \quad \text{as} \quad z \to 0 \quad (2)$$

$$y \to Bz^q \quad \text{as} \quad z \to \infty \quad (3)$$

with $q > p$, the expression

$$y = [(Az^p)^n + (Bz^q)^n]^{1/n} \quad (4)$$

is usually suitable for interpolation. (The variables y and z and hence the coefficients A and B will be considered dimensionless for convenience although this restriction is not necessary.) The right side of Equation (4) can be interpreted as the nth-order sum of the two asymptotic solu-

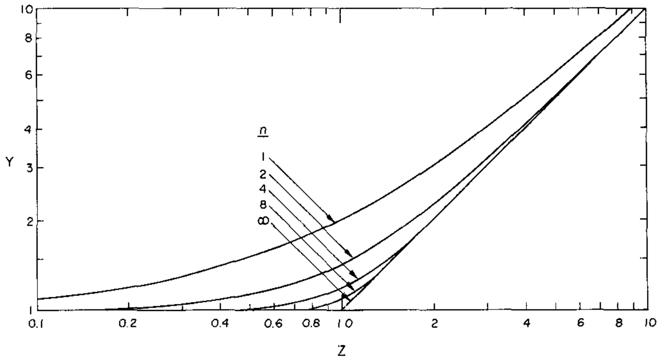

Fig. 1. General functional relationship for transfer processes.

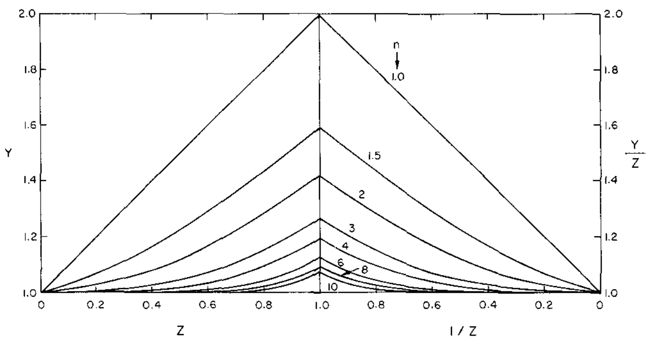

Fig. 2. Working plot for correlation.

tions or as $2^{1/n}$ times the nth-root mean value, for example, if $n = 2$ as $\sqrt{2}$ times the root mean square value. Equation (4) can be rearranged in the two more convenient forms

$$y/Bz^q = [1 + (A/Bz^{q-p})^n]^{1/n} \tag{5}$$

and

$$y/Az^p = [1 + (Bz^{q-p}/A)^n]^{1/n} \tag{6}$$

both of which have the general form

$$Y = [1 + Z^n]^{1/n} \tag{7}$$

Evaluation of n. The exponent n can be evaluated from a single intermediate value of $y(z)$ or $Y(Z)$. If Y is known for $Z = 1$

$$n = \ln(2)/\ln Y(1) \tag{8}$$

For any other value of Z, n must be calculated indirectly. The best value of n for a series of values of $Y(Z)$ could be calculated by nonlinear regression. However, Equation (7) is relatively insensitive to n as indicated by Figure 1 in which log Y is plotted versus log Z for a series of integral values of n. The known experimental or theoretical values can therefore be plotted in Figure 1 and a reasonable value of n chosen by inspection. Alternatively the plotted values can be extrapolated or interpolated to yield $Y(1)$ so that n can be calculated from Equation (8). A plot in the form of Figure 2 which has Y/Z as an ordinate and $1/Z$ as an abscissa for $Z > 1$ is more convenient and accurate for the determination of n since it permits expansion of the scale of the ordinate. Such a working plot can be readily constructed for just the observed range of values of Y and Z. Figure 2 also demonstrates the reciprocal symmetry of Equations (5) and (6) about $Z = Bz^{q-p}/A = 1$. In both Figures 1 and 2, the point $(Y = 1, Z = 1)$ corresponds to the intersection of the lines representing the two limiting solutions. $Y(1) - 1 = 2^{1/n} - 1$ represents the maximum fractional deviation of Y from the limiting solutions. $Z = 1$ represents the dividing line between large and small values of the independent variable. Hence if a single calculation or experiment is to be carried out, it should be at or near $Z = 1$ since n is most sensitive to $Y(1)$. Even if Equation (7) were not used for correlation this value of Y would provide a useful bound for the process.

Example 1. The experimental data for laminar flow through packed beds of spheres are well represented by the Kozeny-Carman equation

$$\Phi = 150\, Re_p \tag{9}$$

and the data for turbulent flow by the Burke-Plummer equation

$$\Phi = 1.75\, Re_p^2 \tag{10}$$

The corresponding value of Y is then $\Phi/150\, Re_p$ and of Z is $1.75\, Re_p/150 = Re_p/87.5$. The central value of Re_p at $Z = 1$ is thus 87.5. Figure 3 is a plot in the form of Figure 2 of a sampling of the data used by Ergun (1952) to develop his correlation. These points scatter widely but are represented on the mean by the straight lines corresponding to $n = 1$. This value yields the Ergun equation

$$\Phi = 150\, Re_p + 1.75\, Re_p^2 \tag{11}$$

These same experimental values are plotted in Figure 4 in the form of Figure 1. This plot would have exactly the same form as the original plot prepared by Ergun to illustrate the success of his correlation if the ordinate were multiplied by 150 and the abscissa by 87.5. Comparison of Figures 3 and 4 confirms the greater sensitivity of the former. Ergun arrived at Equation (11) by heuristic reasoning and then tested his postulate in the form of Figure 4. On the other hand, the data suggest an exponent of n without any postulation when plotted in the form

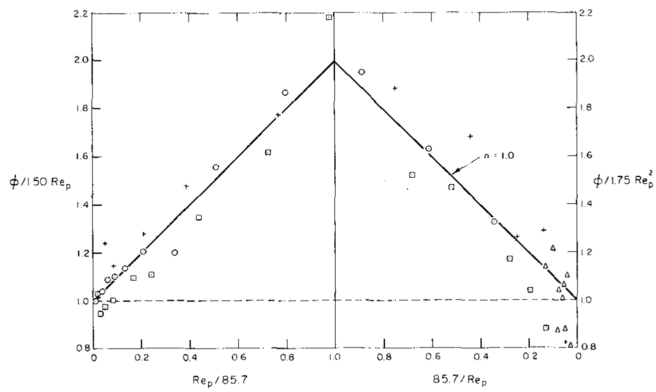

Fig. 3. Pressure drop in flow through a packed bed of spheres.

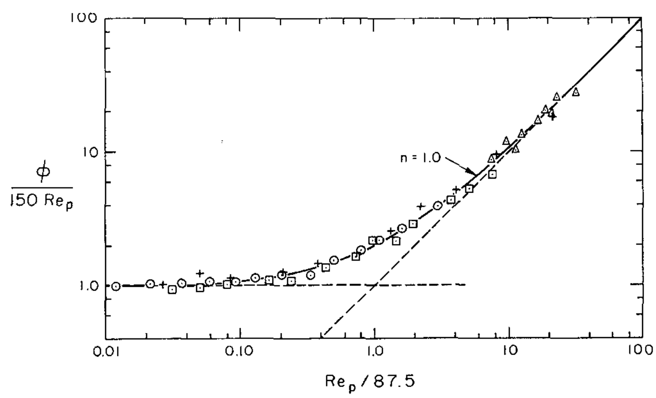

Fig. 4. Logarithmic correlation for pressure drop through a packed bed of spheres.

of Figure 3. In this instance the exponent n was known from the prior work of Ergun. In the several dozen other processes which have been examined the best value of the exponent was not known in advance. This process also happens to be the only one examined so far in which the best exponent is unity.

Example 2. The drag coefficient for a solid sphere falling at low velocity in an incompressible fluid is represented by Stokes' law

$$C_D = 24/Re_d \tag{12}$$

As Re_d approaches 3000 the drag coefficient approaches a constant value of approximately 0.40. (At still higher velocities the drag coefficient has a more complex dependence on velocity.) The experimental data of Christiansen (1943) which are plotted in Figure 5 with $Y = C_D Re_d/24$ and $Z = 0.4\ Re_d/24 = Re_d/60$ are reasonably well represented by Equation (7) with $n = 0.56$ or say 5/9. It is not certain whether the values for $60/Re < 0.4$ are in slight error or whether the physical behavior cannot be represented any more closely by such a simple relationship with constant n. The correlation

$$C_D = (24/Re_d)(1 + [Re_d/60]^{5/9})^{9/5} \tag{13}$$

may be used for $0 < Re_d < 3000$. The central value of Re_d is 60. Drag coefficients for particles of different shapes, both supported in a fluid stream and falling, have a qualitatively similar dependence on Re and could obviously be correlated in the form of Equation (13) with slightly different values of n and the appropriate coefficients substituted for 24 and 60.

Case B. Decreasing Dependence

When the power of the dependent variable is lower at the higher limit, that is, when $p > q$ in Equations (2) and (3)

$$y = [(Az^p)^{-n} + (Bz^q)^{-n}]^{-1/n} \tag{14}$$

can be used instead of Equation (4) for interpolation. Equation (14) can be rearranged as

$$Bz^q/y = [1 + (B/Az^{p-q})^n]^{1/n} \tag{15}$$

or

$$Az^p/y = [1 + (Az^{p-q}/B)^n]^{1/n} \tag{16}$$

which both have the same form as Equation (7). Hence plots such as Figures 1 and 2 can be used for the determination of n in this case as well. (Alternately, Case B can be considered the same as Case A but with a negative value of n.)

The processes considered by Acrivos (1961) were all in this category and he noted that both asymptotic solutions were upper bounds for the dependent variable y.

Example 3. The limiting solutions obtained by Le Fevre (1956) for laminar free convection from an isothermal vertical plate are, as indicated by Equation (1)

$$Nu_x = 0.6004\ Gr^{1/4}Pr^{1/2} \quad \text{for} \quad Pr \rightarrow 0 \tag{17}$$

$$Nu_x = 0.5027\ Gr^{1/4}Pr^{1/4} \quad \text{for} \quad Pr \rightarrow \infty \tag{18}$$

Hence in the form of Equation (15) $Y = 0.5027\ Gr^{1/4}Pr^{1/4}/Nu_x$, $Z = (0.492/Pr)^{1/4}$ and the central value of $Pr = 0.492$. Values obtained from the numerical solutions of Ostrach (1953), Sparrow and Gregg (1959), Sugawara and Michiyoshi (1953) and Saunders (1939) are plotted in Figure 6. A value of $n = 9/4$ represents these values remarkably well yielding

$$Nu_x = 0.503\ Gr^{1/4}Pr^{1/4}/[1 + (0.492/Pr)^{9/16}]^{4/9} \tag{19}$$

The agreement of the values of Ostrach and of Sparrow and Gregg with each other and with the curve for $n = 9/4$ over the entire range suggests that it is the isolated values of Saunders and of Sugawara and Michiyoshi which are in error. The fact that these latter deviations are less than 1% emphasizes the extreme sensitivity of this plot. The better representation is at least in part due to the greater precision of these computed values as contrasted with the use of experimental data in Figures 3 and 5. Surprisingly, Equations (1) and (19) agree to within about 0.1% over the entire range. Ede (1967) indicates that Equation (1) is central to the widely scattered experimental data. Equation (19) can, therefore, also be considered to represent the experimental data.

This was one of the processes considered by Acrivos (1961). He proposed $n = 2$ for representation of the computed values of Ostrach (1953), presumably on the basis of a log-log plot of $1/Y$ versus Z. The superiority of the coordinates of Figure 2 in general and of Figure 6 specifically is thus substantiated in that $n = 9/5$ is shown decisively to be a better choice than $n = 2$, even though the latter value is acceptable for practical purposes.

Case C. Nonpower Dependence

The above derivations are all for processes in which the rate depends on the independent variable to a fixed power in the limits. Many transfer processes fall in this category. However, the usefulness of Equation (7) for interpolation is not limited to such functional relationships. Equation (5) can be generalized to the form

$y(z)/y(z \rightarrow \infty)$

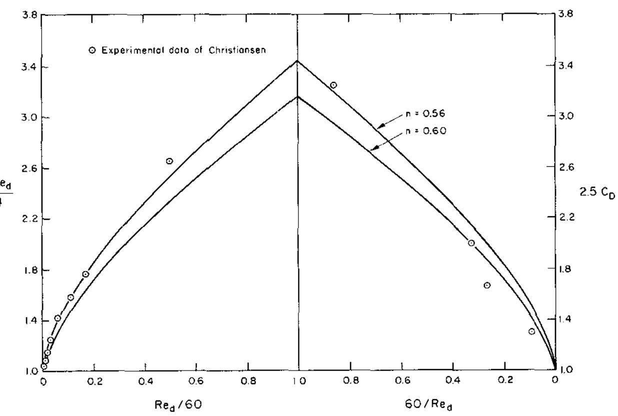

Fig. 5. Drag coefficient for a freely falling sphere.

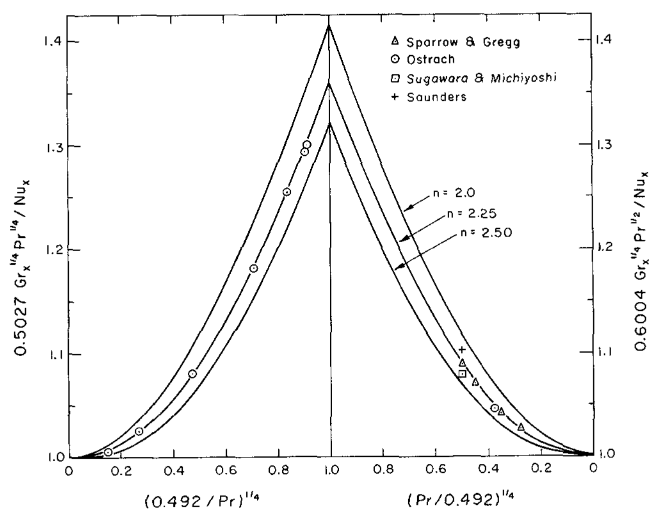

Fig. 6. Laminar free convection from a vertical isothermal plate.

$$= [1 + (y(z \to 0)/y(z \to \infty))^n]^{1/n} \quad (20)$$

where $y(z \to 0)$ and $y(z \to \infty)$ are the asymptotic solutions for $z \to 0$ and $z \to \infty$, respectively. Equations (6), (15), and (16) can be generalized similarly.

Example 4. Transient conduction through a finite layer of insulation to a semi-infinite region of lesser conductivity following a step change in the external surface temperature with both the insulation and the region at a uniform initial temperature, as discussed by Churchill (1962), can be used to illustrate this more general case. The asymptotic solutions for the heat flux density at the external surface are

$$j(\pi\, \alpha t)^{1/2}/k(T_w - T_\infty) = 1 \quad \text{as} \quad t \to 0 \quad (21)$$

and

$$j\, \delta/k(T_w - T_\infty) = e^{\alpha t/\sigma^2\delta^2}\, \text{erfc}\, (\alpha\, t/\sigma^2\delta^2)^{1/2} \quad \text{as} \quad t \to \infty \quad (22)$$

When plotted in the form suggested by Equations (20), (21), and (22) the intermediate values computed from the series solution for $\sigma = 49$ suggest a value of $n = 8$, yielding

$$j(\pi\, \alpha t)^{1/2}/k(T_w - T_\infty) = (1 + [(\pi\, \alpha t/\delta^2)^{1/2}\, e^{\alpha t/\sigma^2\delta^2}\, \text{erfc}(\alpha t/\sigma^2\delta^2)^{1/2}]^8)^{1/8} \quad (23)$$

Equation (23) agrees with the computed values to four significant figures over the entire range as indicated in Figure 7. Analysis suggests that n will become a significant function of σ for $\sigma < 20$. Hence a plot of n versus σ would complete the correlation. The central value of time corresponding to the intersection of the dashed line and dashed curve in Figure 7 or to the simultaneous solution of Equations (21) and (22) is approximately $\delta^2/\pi\alpha$.

Case D. Ambiguity in the Choice of the Asymptotic Solution

Churchill and Ozoe (1972) developed correlations for laminar forced convection in flow over flat plates and through tubes. Their work for fully developed flow through tubes illustrates the process of analysis which may be required to choose the appropriate asymptotic solutions.

Example 5. Following a step function in wall temperature in fully developed laminar flow in a tube Nu_D approaches 3.657 as $Gz \to 0$. For the other limit Lévêque (1928) derived a solution

$$Nu_D = 1.167\, Gz^{1/3} \quad \text{for} \quad Gz \to \infty \quad (24)$$

Lipkis (1955) re-expressed Equation (24) in terms of the mixed-mean temperature rather than the inlet temperature. The resulting equation

$$Nu_D = 1.167\, Gz^{1/3}/[1 - 5.5/Gz^{2/3}] \quad (25)$$

would be expected to be applicable to lower values of Gz than Equation (24). However, a correlation based on Equation (25) would retain the singularity at $Gz \cong 13.0$. Hence, Churchill and Ozoe (1972) used Equation (24) rather than Equation (25) to derive the following expression for interpolation

$$Nu_D/3.657 = [1 + (Gz/30.8)^{8/3}]^{1/8} \quad (26)$$

Equation (26) represents the values obtained from the Graetz series and by direct numerical integration of the differential energy balance reasonably well. However, the computed points fall slightly below unity for large Gz indicating that the computations are in error or that the Lévêque solution is not a true lower bound.

The solution obtained by Worsøe-Schmidt (1967) as an expansion about the Lévêque solution confirms that the

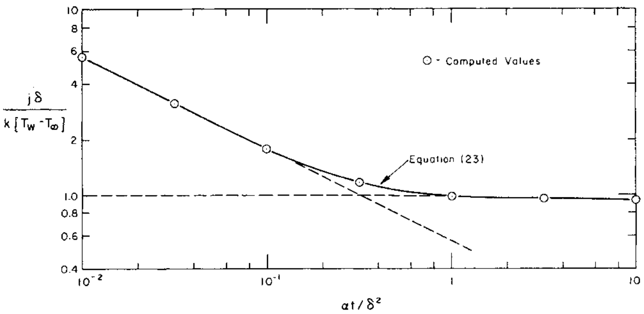

Fig. 7. Conduction to an insulated, semi-infinite region after a step in surface temperature $\sigma = 49$

Lévêque solution is not a lower bound for the general solution and hence is not satisfactory on theoretical grounds for the construction of a correlation. The results of Worsøe-Schmidt for large Gz can be approximated by the expression

$$Nu_D = 1.167\, Gz^{1/3} - 1.7 \quad (27)$$

as suggested by Lipkis (1955). If the right side of Equation (27) is used as the asymptotic solution for large Gz the general correlation takes the form

$$Nu_D/3.657 = (1 + [(Gz/30.8)^{1/3} - 0.465]^n)^{1/n} \quad (28)$$

Equation (28) is clearly limited to $Gz > 3.10$ in order to avoid negative values for the term in square brackets. This is not a serious limitation since Nu_D can simply be taken as 3.657 without significant error for $Gz < 3.10$. A value of $n = 2$ represents the computed values well for all $Gz > 3.10$; but some dissymmetry is apparent.

Because of these two discrepancies in the representation provided by Equation (28) an alternative correlation was constructed as follows. Equation (27) was rearranged as

$$Nu_D + 1.7 = 1.167\, Gz^{1/3} \quad (29)$$

The corresponding expression for correlation is then

$$(Nu + 1.7)/5.357 = [1 + (Gz/97)^{n/3}]^{1/n} \quad (30)$$

Again in this form the computed values do not demonstrate the symmetry found previously in Examples 1 through 4. Apparently the appearance of an additive term in one of the asymptotic solutions precludes the symmetry implied by Equation (7). However $n = 8/3$ represents the computed values well for all Gz.

These examples illustrate a limitation on the ability of Equation (7) to represent this particular transfer process. However, the limitation is of greater intrinsic than practical interest. Equation (26) and Equations (28) and (30) with a single compromise value of n provide reasonably good correlations (within 5%) over the entire range of Gz from 0 to ∞. Correlations within the range of certainty of the most reliable sets of computed values ($\sim 1\%$) are provided by both Equations (28) and (30) by using different values of n for large and small Gz. Equation (26) is even more accurate for all but the short range of Gz in which the additive term in Equation (27) is influential. All of these correlations have a greater range of applicability than the Worsøe-Schmidt series solution and are more convenient to use than the Graetz series solution or the various sets of tabulated values. The sensitivity of this method of correlation is again confirmed in that the failure of the Lévêque solution by a very slight amount to provide a lower bound was revealed in the plot which led to Equation (26).

A similar discrepancy was revealed in the development of a correlation for steady heat and component transfer from a sphere in Stokes flow by Brian and Hales (1969). They correlated their values with an equation equivalent to Equation (7) but determined the coefficient in the asymptotic solution for large flow rates empirically. This coefficient was observed to be 10% higher than that of the asymptotic solution obtained analytically after several approximations. Their calculations were sufficiently precise to reveal this discrepancy in a log-log plot of the original variables. A plot in the form of Figure 2 would probably have revealed the discrepancy even with less precise values.

Case E. Multiple Variables

The effect of secondary variables can be taken into account by determining n for a series of values of the second variable and developing a graphical correlation for n as suggested in Example 4. In some instances the use of Equation (20) can be repeated in series for two or more variables. This process is illustrated below.

Example 6. Churchill and Ozoe (1972) obtained an excellent representation for their computer values and those of prior workers for laminar forced convection in flow over an isothermal plate with the expression

$$Nu_x = 0.564\, Re_x^{1/2} Pr^{1/2} / [1 + (Pr/0.0468)^{2/3}]^{1/4} \quad (31)$$

A value of n less than 4 would represent the computed values slightly better for large Pr and a value greater than 4 for small Pr, but Equation (31) agrees to approximately 1% with all of the computed values.

Acrivos (1961) recommended $n = 3$ for all wedge flows including the flat plate. Analysis of his computed values (which he graciously supplied) in the form of Figure 2 indicates that $n = 3$ provides an excellent fit for flow normal to the plate but reveals that the best value of n increases to 4 with decreasing wedge angles.

Equation (31) was reinterpreted and rearranged as an asymptotic solution for $Gz \to \infty$ for forced convection in developing laminar flow in an isothermal tube in the following form:

$$Nu_D = 0.637\, Gz^{1/2} / ([1 + (Pr/0.0468)^{2/3}]^{1/4} - 4Gz^{-1/2}) \quad (32)$$

Equation (32) might be combined in turn with $Nu_D = 3.657$ for fully developed heat transfer to yield an expression for all Gz and all Pr in developing flow. However, it is apparent that the resulting expression would blow up for $Gz \leq 16/[1 + (Pr/0.0468)^{2/3}]^{1/2}$. Hence the term $4Gz^{-1/2}$ in Equation (32) was dropped for this application just as was the comparable term in Equation (25) since these terms become negligible anyway as $Gz \to \infty$. The resulting combined expression with a value of $n = 9/5$ was found to yield a reasonable representation for the several sets of computed values which are available for $Pr \leq 2.0$. The choice of n and the success of the form of the correlation is subject to considerable uncertainty because the computed values scatter widely, and it is not apparent which are the most reliable and to what degree. However, this correlation yields values for Uu_D for $Pr \to \infty$ below those given by the Graetz solution, which is presumed to be a lower bound for developing flow. Hence, the reduced form of Equation (20) was combined instead with Equation (26) to yield

$$\frac{Nu_D}{3.657\,[1 + (Gz/30.8)^{8/3}]^{1/8}} = \left[1 + \left(\frac{(Gz/33)^{1/2}}{[1 + (Pr/0.0468)^{2/3}]^{1/4}\,[1 + (Gz/30.8)^{8/3}]^{1/8}}\right)^n\right]^{1/n} \quad (33)$$

Equation (33) with $n = 4$ was found to represent the computed values equally well and behaves correctly in all of the chosen limiting cases. For example for $Pr \to 0$ and $Gz \to \infty$ Equation (33) appears to converge to the Graetz solution for plug flow which is presumed to be an upper bound for developing flow. Again the previously mentioned anomalies in the computed values provide some uncertainty in the choice of n. Furthermore, the computed values for very large Gz are consistently less than predicted by Equation (33) and indeed by Equation (32). This discrepancy indicates that Equation (32) is not truly a lower bound for developing flow in tube just as Equation (25) is not for developing flow.

The value of 1.7 was therefore arbitrarily added to the left side of Equation (32). This expression without the $4Gz^{-1/2}$-term was then combined with Equation (30) with 8/3 to yield

$$\frac{Nu_D + 1.7}{5.357\,[1 + (Gz/97)^{8/9}]^{3/8}} = \left[1 + \left(\frac{(Gz/71)^{1/2}}{[1 + (Pr/0.0468)^{2/3}]^{1/4}\,[1 + (Gz/97)^{8/9}]^{3/8}}\right)^n\right]^{1/n} \quad (34)$$

Equation (34) appears to behave correctly in all of the limiting cases of large and small Gz and Pr. $n = 8/3$ represents the computed values for all Gz reasonably well. The additive term does not appear to introduce as much asymmetry as it did in the case of Equation (30). Some of the computed Nusselt numbers are less than the values given by Equation (30) which certainly should be a lower bound. However, these deviations and other wide divergences are probably due to computing error rather than to the inadequacy of the correlation.

The discrepancies in these several correlations for forced convection in developing flow arise not from the application of Equation (20) in series for two variables but rather from the limitations in the component equations. Equations (33) and (34) appear somewhat ungainly but are probably the simplest possible expressions that converge to all four of the chosen asymptotic solutions.

INTERPRETATION

In all of the above processes a uniform transition occurs between the limiting values. Clearly Equation (7) cannot represent processes which have an irregular transition such as from laminar to turbulent flow in pipes. A sufficient restriction to permit the use of Equation (7) is that the first derivative of the functional relationship be continuous and that the second derivative not change sign. This restriction can be relaxed somewhat as indicated by the success achieved in example 4 in which the second derivative changes sign, but slowly.

Symmetry about $Z = 1$ is not necessarily to be expected for all transfer processes, that is, the same value of n would not be expected to represent best both the data for Z greater and less than unity. Indeed the functional relationship itself provided by Equation (7) cannot be justified theoretically for either Z greater or less than unity for the inherently complex transfer processes. The remarkable agreement in functional form in most of the above examples and the relative insensitivity to the value of n demonstrated in some of the examples above must therefore be considered somewhat fortuitous. The maximum deviation of the correlation from the asymptotic solutions is $2^{1/n} - 1$. Hence the larger the value of n the smaller the range of the correlation. The success of Equation (7) in correlating diverse processes is related to this constraint.

Some care must be taken in the choice of the proper asymptotic solution. A singularity which is not of concern in an asymptotic solution may be unacceptable in the combined expression. The chosen asymptotic solutions should be lower bounds in Case A (increasing dependence) and upper bounds in Case B (decreasing dependence) to avoid anomalies in the combined expression. Additive terms in the asymptotic solutions appear to produce asymmetry in the correlation and it may be necessary to choose different exponents for large and small values of Z to provide a precise correlation in such cases. The asymptotic solutions yielding the largest value of n should be chosen if possible since this reduces the range of $Y(z)$. Thus the use of Equation (26) instead of the limiting value of 3.657 yielded $n = 4$ as opposed to $n = 1.8$ in Example 6. Equation (26) with $n = 8$ is competitive with Equation (28) with $n = 2$ and with Equation (30) with $n = 8/3$ for this reason despite its theoretical shortcomings. The previously discussed experience of Brian and Hales (1969) suggests that the coefficient in one or both of the asymptotic solutions might be determined by trial and error if the values are uncertain.

The central value of the independent variable is defined by the limiting solutions and is not a function of n. It is noteworthy that the central value of Pr is 0.492 in Example 3 but 0.0468 in Example 6. Similarly the central value of Gz is 30.8 in Equation (26) but depends on Pr in Equation (33). Thus the central value of the independent variable depends on the process itself. This central value should be given priority in experimentation or in a computational program since it leads most directly and most accurately to the best value of n.

Expressions having a form equivalent to Equation (7) have been used for correlation in many instances in the past, for example by Sparrow (1955) and Brian and Hales (1969). The generality acquired by the definition of Y and Z in terms of the appropriate asymptotic solutions has received more limited attention, notably from Acrivos (1961) for the effect of Pr and Sc on convection in boundary layer flows. However, he did not discuss the evaluation of n or the applicability of the expression to other processes.

Equation (7) and Figures 1 and 2 can be considered to be approximate representations for all phenomena in which the dependent variable varies uniformly from small to large values of the independent variable. The variables Y and Z are defined by the limiting solutions for large and small values of the independent variable and the individual processes are characterized by the choice of n alone. Indeed a value for n is a sufficient documentation for the correlation if the choice of the limiting solutions is unambiguous.

In summary Equation (7) and the procedure described above appear to have wide applicability for correlating or testing experimental or theoretical values for processes in which the limiting behavior is known. The examples above were for simple processes of flow and heat transfer but the procedure is equally applicable to component transfer, to more complex processes of transfer and indeed to the correlation of thermodynamic data, etc. Processes which produce anomalies were deliberately chosen for illustration to indicate the limitations as well as the successes of the equation and procedure.

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NOTATION

A = coefficient in asymptotic solution for $z \to 0$ [see Equation (2)]
B = coefficient in asymptotic solution for $z \to \infty$ [see Equation (3)]
c = heat capacity, J/kg·K
c_D = drag coefficient for sphere
d_p = diameter of sphere, m
D = diameter of pipe, m
D_f = diffusivity, m^2/s
g = gravitational acceleration, m/s^2
Gr = $g\beta(T_w - T_\infty)x^3/\nu^2$ = Grashof number
Gz = wc/kx = Graetz number
h = local heat transfer coefficient, $W/m^2 \cdot K$
j = local heat flux density, W/m^2
k = thermal conductivity of fluid or insulation, W/m·K
k' = thermal conductivity of semi-infinite region, W/m·K
L = distance through packed bed, m
n = exponent in Equation (7)
Nu_x = hx/k = local Nusselt number for plate
Nu_D = hD/k = local Nusselt number for tube
P = pressure, $kg/m \cdot s^2$
p = exponent in asymptotic solution for $z \to 0$ [see Equation (2)]
q = exponent in asymptotic solution for $z \to \infty$ [see Equation (3)]
Re_x = xu_∞/ν = local Reynolds number for plate
Re_d = $d_p u_0/\nu$ = Reynolds number for falling sphere
Re_p = $d_p u_0/\nu(1 - \epsilon)$ = Reynolds number for packed bed of spheres
Sc = ν/D_f = Schmidt number
T_w = temperature of wall, K
T_∞ = ambient temperature, K
u_0 = superficial velocity through packed bed or velocity of sphere, m/s
u_∞ = free-stream velocity, m/s
w = mass flow rate in pipe, kg/s
x = distance from start of plate, m
$y(z)$ = dependent variable
$y(z \to 0)$ = asymptotic behavior of dependent variable for $z \to 0$
$y(z \to \infty)$ = asymptotic behavior of dependent variable for $z \to \infty$
$Y(Z)$ = normalized dependent variable [see Equation (7)]
z = independent variable
Z = normalized independent variable [see Equation (7)]

Greek Letters

α = thermal diffusivity of fluid or insulation, m^2/s
α' = thermal diffusivity of semi-infinite region, m^2/s
β = volumetric coefficient of expansion with temperature, 1/K
δ = thickness of insulation, m
ϵ = void fraction in packed bed
μ = viscosity, kg/m·s
ν = kinematic viscosity, m^2/s
ρ = density, kg/m^3
σ = $(k'/k)(\alpha/\alpha')^{1/2}$
Φ = $\rho\epsilon^3 d_p^3(-dP/dL)/\mu^2(1 - \epsilon)^3$

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Modal Analysis of Convection with Axial Diffusion

The one-dimensional model for convection with diffusion and with a source term for mass or energy generation or interchange is analyzed for the eigenvalues and the corresponding spatial eigenmodes as a function of the Peclet number. It is shown how the modal analysis of the source case, when the source coefficients for heat exchange and chemical reaction are spatially independent, is directly related to the no source solution. Numerical examples of determining the source term coefficient are included. These solutions form a base to discuss dynamic characteristics and stability and to which solutions for spatially dependent coefficients can be compared.

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SCOPE

Eigenvalues and corresponding eigenfunctions arise naturally and are useful in connection with systems described by linear partial differential equations. They are useful in the solution of the equations with given initial conditions and in the determination of system stability, measurement, and control. Linear partial differential equations arise when small perturbations about some operating state are considered for a linear or nonlinear system. In this paper we obtain the eigenvalues and associated eigenfunctions for the one dimensional linearized model for convection with axial diffusion and with a source term for mass or energy generation or interchange subject to the widely used Wehner and Wilhelm (1956) boundary conditions.

$$\frac{1}{Pe}\frac{\partial \phi}{\partial x^2} - \frac{\partial \phi}{\partial x} + \sigma\phi = \frac{\partial \phi}{\partial t}$$

$$\phi_F(t) = \phi(0, t) - \frac{1}{Pe}\frac{\partial \phi}{\partial x}(0, t)$$

$$\frac{\partial \phi}{\partial x}(1, t) = 0$$

This model has wide applicability arising in connection

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